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WILSON'S THEOREM FOR FINITE FIELDS

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ABSTRACT. In this short note, we introduce an analogue of Wilson's theorem for all nonzero elements $a_1, a_2, \cdots, a_{q-1}$ of a finite filed $\mathbb{F}$ with $|\mathbb{F}| = q \geq 3$, as follows:

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq q-1} a_{i_1} a_{i_2} \cdots a_{i_k} = \left\lfloor \frac{k}{q-1} \right\rfloor (-1)^q \qquad (k = 1, 2, \cdots, q-1),$$

which the left hand side of above formula is the $k-$th elementary symmetric polynomial evaluated at $a_1, a_2, \cdots, a_{q-1}$. Specially, letting $\mathbb{F} = \mathbb{Z}_p$ with $p \geq 3$, reproves Wilson's theorem and yields some Wilson type identities. Finally, we obtain an analogue of Wolstenholme's theorem for nonzero elements of a finite filed.

Let $\mathbb{F}$ be a finite filed with $\mathrm{char}(\mathbb{F}) = p$ and set $|\mathbb{F}| = p^n = q$. So, $|\mathbb{F}^*| = q - 1$ where $\mathbb{F}^* = \mathbb{F} - \{0\} = \{a_1, a_2, \cdots, a_{q-1}\}$. By Lagrange's theorem, if $a \in \mathbb{F}^*$ then $o(a)|q-1$ and so $a^{q-1} = 1$ or $a^q = a$. This equation holds also for $a = 0$. Therefore, the elements of $\mathbb{F}$ are the roots of $x^q - x$. However, this polynomial has at most q roots, so the elements of $\mathbb{F}$ are precisely the roots of $x^q - x$. Thus, we obtain:

$$x^q - x = x(x^{q-1} - 1) = x \prod_{i=1}^{q-1} (x - a_i).$$

Note that $a_i \neq 0$ for $i = 1, 2, \cdots, q-1$ and we let $q \geq 3$. Considering elementary symmetric functions [2];

$$s_k = s_k(a_1, a_2, \cdots, a_{q-1}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq q-1} a_{i_1} a_{i_2} \cdots a_{i_k},$$

we have

$$\begin{aligned} x^{q-1} - 1 &= (x - a_1)(x - a_2) \cdots (x - a_{q-1}) \\ &= x^{q-1} - s_1 x^{q-2} + s_2 x^{q-3} + \cdots + (-1)^{q-1} s_{q-1}, \end{aligned}$$

or the following identity:

$$\sum_{k=1}^{q-1} (-1)^k s_k x^{q-1-k} + 1 = 0 \qquad (x \in \mathbb{K}^*),$$

where $\mathbb{K}$ is a field extension of $\mathbb{F}$, and comparing coefficients, we obtain:

$$s_1 = 0, s_2 = 0, \cdots, s_{q-2} = 0 \text{ and } s_{q-1} = (-1)^{q-1} a_1 a_2 \cdots a_{q-1},$$

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which we can state all of them together as follows:

$$s_k = \left\lfloor \frac{k}{q-1} \right\rfloor (-1)^q \qquad (k = 1, 2, \cdots, q-1 \text{ and } q \geq 3).$$

This relation is a generalization of the Wilson's theorem for nonzero elements of a finite field. Specially, letting $\mathbb{F} = \mathbb{Z}_p$ with $p \geq 3$, we obtain:

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p-1} i_1 i_2 \cdots i_k \equiv -\left\lfloor \frac{k}{p-1} \right\rfloor \qquad (\text{mod } p),$$

where $k = 1, 2, \cdots, p-1$ and putting $k = p-1$, it reproves Wilson's theorem [1]; $(p-1)! \equiv -1 \pmod{p}$.
Moreover, above mentioned results assert us to calculate the summations

$$\sum_{k=1}^{q-1} \Phi(a_1, a_2, \cdots, a_{q-1}),$$

where Φ is a symmetric function of $a_1, a_2, \cdots, a_{q-1}$; because for above given Φ, there exists the function Ψ, such that:

$$\Phi(a_1, a_2, \cdots, a_{q-1}) = \Psi(s_1, s_2, \cdots, s_{q-1}) = \Psi(0, 0, \cdots, 0, s_{q-1}).$$

For example

$$\sum_{k=1}^{q-1} \frac{a_1 a_2 \cdots a_{q-1}}{a_k} = (-1)^{q-2} s_{q-2} = 0 \qquad (q \geq 5),$$

which is an analogue of Wolstenholme's theorem [1]; $\sum_{k=1}^{p-1} \frac{(p-1)!}{k} \equiv 0 \pmod{p^2}$ and $p \geq 5$.

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References

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[2] Patrick Morandi, *Field and Galois Theory*, Graduate Texts in Mathematics, Vol. 167, Springer-Verlag, New York, 1996.

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